Hamiltonian Structures of KdV-Type Hierarchies and Associated W-Algebras

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ABSTRACT

The $(n,m)^{\mathrm{th}}$ KdV hierarchy is a restriction of the KP hierarchy to a submanifold of pseudo-differential operators in a radio form. Explicit formula of the restricted Hamiltonian structure of KP is given which provides a new, more constructive proof of the isomorphism between the associated $W(n,m)$-algebra to $W_{n+m} \oplus W_m \oplus U(1)$ algebra, and the Hamiltonian property of the $(n,m)^{\mathrm{th}}$ KdV hierarchy as well as its Lax-Manakov triad representation. Similarly the Hamiltonian property for a version of modified n^{th} KdV and the isomorphism between W_n-algebra to $W_l \oplus W_m \oplus U(1)$ algebra are shown, where $l+m=n$. The role of $U(1)$ current in both cases is also explained.

Recently, Bonora et al. [1,2] proposed a KdV-type integrable system, which they called the $(n, m)^{\text{th}}$ KdV hierarchy. This hierarchy is obtained by restricting the n^{th} order pseudo-differential operators (PDOs) associated with the KP hierarchy to a submanifold of PDOs of the following form

$$L = \mathrm{e}^{-m\theta} A \mathrm{e}^{-n\theta} B^{-1} \mathrm{e}^{(n+m)\theta}, \tag{1}$$

where $m \geq 2$, A and B are two pure differential operators of the $(n+m)^{\text{th}}$ and m^{th} order respectively and both without the second leading terms. Thus the hierarchy contains equations for the $n + 2m - 1$ independent unknowns: θ (or $J = \theta_x$), coefficients of A and B. They showed the compatibility of the restriction with the KP hierarchy, displayed the first nontrivial flow and most importantly, they proved that the so-called $W(n, m)$-algebra appearing as the second Poisson bracket of the $(n, m)^{\text{th}}$ KdV hierarchy is isomorphic via a Miura map to the direct sum of the W_{n+m}-algebra and W_m-algebra, as well as an additional $U(1)$ current algebra. As was shown in [2], the proof of this isomorphism depends on the derivation of the first nontrivial Hamiltonian equation and the uniqueness of the second Poisson structure of the equation. In [3] Dickey showed another viewpoint on this hierarchy and provided a more constructive proof of compatibility of such restriction with the KP hierarchy.

Despite of these studies and a closely connection of the $(n, m)^{\text{th}}$ KdV hierarchy and the so-called constrained KP hierarchy given in a series papers (see [4-8]), the structure of all equations in the $(n, m)^{\text{th}}$ KdV hierarchy, particularly their Hamiltonian properties still need the further inverstigation.

In this letter we first show the restriction of the Poisson bracket of the KP hierarchy and show the explicit formula of the Poisson bracket in terms of the finite numbers of coordiantes (or called fields) θ, and coefficients of A and B. As results we are able to present an exact and pure algebraic proof of the isomorphism between $W(n, m)$-algebra and $W_{n+m} \oplus W_m \oplus U(1)$ algebra, and show that all the equations in the $(n, m)^{\text{th}}$ KdV hierarchy are Hamiltonian and possess the Lax-Manakov triad representations.

Then we immediately reconganize that the above results can be generalized to the n^{th} KdV hierarchy (or called Gelfand-Dickey hierarchy). If we restrict the n^{th} pure differential operator L by

$$L = \mathrm{e}^{-m\theta} A \mathrm{e}^{n\theta} B \mathrm{e}^{-l\theta}, \tag{2}$$

with $l + m = n$, $2 \leq l, m \leq n - 2$, and A and B being respectively of the l^{th} and m^{th} order of pure differential operators both without the second leading terms, then we have the parallel results, namely we can prove that the W_n-algebra associated with the second Hamiltonian structure of the n^{th} KdV hierarchy is isomorphic via a Miura map to the direct sum of a

W_l-algebra, a W_m-algebra and a $U(1)$ current algebra, and we have a Hamiltonian system and the Lax-Manakov triad representation for $J = \theta_x, A, B$. Euqations in this system can be considered as a version of the modified equations of the n^{th} KdV equations.

In both cases of $W(n,m)$-algebra and W_n-algebra, the $U(1)$ current J plays an important role. We explain its role from the viewpoint of free fields realization and prove that it is a conserved dentity of the $(n,m)^{\text{th}}$ KdV hierarchy and the modified n^{th} KdV hierarchy for J, A, B.

In general let

$$\mathcal{L} = \partial^n + \sum_{j=-\infty}^{n-1} u_j \partial^j, \tag{3}$$

be the PDO of order n. The bi-Hamiltonian structures can be built on the space of such type PDOs, in particular the second Poisson bracket is given by (see [9])

$$\{\tilde{f}, \tilde{g}\}^{(n)} = \int \text{res} \left(\left(\mathcal{L} \frac{\delta f}{\delta \mathcal{L}} \right)_+ \mathcal{L} - \mathcal{L} \left(\frac{\delta f}{\delta \mathcal{L}} \mathcal{L} \right)_+ \right) \frac{\delta g}{\delta \mathcal{L}} dx, \tag{4}$$

where $\tilde{f} = \int f(u_{n-1}, u_{n-2}, \cdots) dx$ with $f(u_{n-1}, u_{n-2}, \cdots)$ being differential polynomial in $u_{n-1}, u_{n-2}, \cdots$,

$$\frac{\delta f}{\delta \mathcal{L}} = \sum_{r=-\infty}^{n-1} \partial^{-r-1} \frac{\delta f}{\delta u_r}, \tag{5}$$

and similarly for $\tilde{g}$ and $\delta g / \delta \mathcal{L}$. Here and after for a PDO $A = \sum_{i \le n} a_i \partial^i$, $\text{res} A = a_{-1}$ and $A_\pm$ denote its differential and residual parts respectively.

If we take the reduction of the Poisson bracket (4) to the submanifold of $u_{n-1} = 0$, the term $\delta f / \delta u_{n-1}$ in $\delta f / \delta \mathcal{L}$ becomes indefinite, so the following condition

$$\text{res} \left[\frac{\delta f}{\delta \mathcal{L}}, \mathcal{L} \right] = 0 \tag{6}$$

should be teken into account such that $\delta f / \delta u_{n-1}$ is eliminated from (2) by expressing it in terms of other coefficients $\delta f / \delta u_r$ [9]. By this condition we have

Proposition 1. *The Poisson bracket reduced to the submanifold $u_{n-1} = 0$ is in the form*

$$\begin{aligned} \{\tilde{f}, \tilde{g}\}_R^{(n)} = & \int res \left(\left(L \frac{\delta f}{\delta L} \right)_+ L - L \left(\frac{\delta f}{\delta L} L \right)_+ \right) \frac{\delta g}{\delta L} dx \\ & - \frac{1}{n} \int \left(res \left[L, \frac{\delta g}{\delta L} \right] \left(\partial^{-1} res \left[L, \frac{\delta f}{\delta L} \right] \right) \right) dx, \end{aligned} \tag{7}$$

where

$$L = \partial^n + \sum_{j=-\infty}^{n-2} u_j \partial^j, \tag{8}$$

and

$$\frac{\delta f}{\delta L} = \sum_{r=-\infty}^{n-2} \partial^{-r-1} \frac{\delta f}{\delta u_r}. \tag{9}$$

Equation (7) was appeared in [1] as well as in [7] for a special case, however it still needs a more general proof. We display such proof below.

Proof. Let

$$X = \sum_{r=-\infty}^{n-1} X_r = \partial^{-n} X_{n-1} + \tilde{X},$$

the conditions $u_{n-1} = 0$ and (6) lead to

$$\text{res}[\mathcal{L}, X] = \text{res}[L, \partial^{-n} X_{n-1}] + \text{res}[L, \tilde{X}] = 0,$$

from which we have

$$n \frac{\partial X_{n-1}}{\partial x} + \text{res}[L, \tilde{X}] = 0. \tag{10}$$

Substituting $X = \delta f / \delta \mathcal{L} = \partial^{-n} X_{n-1} + \tilde{X}$, $\tilde{X} = \delta f / \delta L$ and the similar expression for $Y = \delta g / \delta \mathcal{L}$ into (4) and noting that

$$(XL)_+ = X_{n-1} + (\tilde{X} L)_+, \quad (LX)_+ = X_{n-1} + (L\tilde{X})_+,$$

we may directly have the result.

In terms of the fields $\{u_j\}$, the Poisson brackets among themself are given by

$$\{u_r(x),\ u_s(y)\}_R^{(n)} = J_{r,s}^{(n)}(u)\delta(x-y), \tag{11}$$

for $r, s = n-2, \cdots$ and is called the $\hat{W}_\infty^{(n)}$-algebra (see [10]), it contains the following Virasoro algebra $Vir[c(n)]$

$$\{u_{n-2}(x), u_{n-2}(y)\}_R^{(n)} = -\left(\frac{c}{12}\partial^3 + u_{n-2}\partial + \partial u_{n-2}\right)\delta(x-y), \tag{12}$$

as its subalgebra, where

$$c(n) = (n-1)n(n+1) \tag{13}$$

is the central charge.

It is well-known (see [9]) that the flows in the KP hierarchy are Hamiltonian with respect to the Poisson bracket (7) and the Hamiltonians are

$$\tilde{h}_r(L) = \frac{n}{r} \int \mathrm{res} L^{\frac{r}{n}} dx, \quad r = 1, 2, \cdots \tag{14}$$

These quantities commute with each other. The r^{th} flow is written as

$$\begin{aligned}
\partial_{t_r} L &= \sum_{j=-\infty}^{n-2} (\partial_{t_r} u_j)\, \partial^j \\
&= \sum_{j=-\infty}^{n-2} \{\tilde{h}_r, u_j\} \partial^j \\
&= \left(L \frac{\delta h_r}{\delta L} \right)_+ L - L \left(\frac{\delta h_r}{\delta L} L \right)_+ .
\end{aligned}$$

Since

$$\frac{\delta h_r}{\delta L} = L^{\frac{r-n}{n}} \mod R(-\infty, -n-1), \tag{15}$$

where $R(i,j)$ denotes the set of all PDOs of the form $P = \sum_{r=i}^{j} P_r \partial^r$, the r^{th} flow can be written as the following Lax representation

$$\partial_{t_r} L = P_r L - L P_r, \tag{16}$$

with

$$P_r = \left(L^{\frac{r}{n}} \right)_+ . \tag{17}$$

When L is in the subspace consisting of pure differential operators of order n, (16) represents the n^{th} KdV hierarchy and (7) its second Poisson bracket. The correspondent Poisson algebra (11) is called W_n-algebra, where the subscript runs from $n-2$ to 0.

Now we restrict the PDO in the following form

$$L = \mathrm{e}^{-m\theta} A \mathrm{e}^{\alpha n \theta} B^{\alpha} \mathrm{e}^{-\alpha l \theta}, \tag{18}$$

where

$$\alpha = \pm 1, \quad l = n - \alpha m, \tag{19}$$

with $2 \le m \le n-2$, and A and B are two pure differential operators with the order of l and m respectively

$$\begin{aligned}
A &= \partial^l + v_{l-2} \partial^{l-2} + \cdots + v_0, \\
B &= \partial^m + w_{m-2} \partial^{m-2} + \cdots + w_0.
\end{aligned} \tag{20}$$

For $\alpha=-1$ or $\alpha=1$, (18) is the restriction (1) or (2).

So we have a group of new fields θ (sometime we use $J=\theta_x$), $v_{l-2},\cdots,v_0$ and $w_{m-2},\cdots,w_0$. The Miura map between the fields $u_{n-2},\cdots,$ and the new fields can be obtained by comparing coefficients in (18). According to these new fields we have

Porposition 2. *The restriction of (18) leads the Poisson bracket in (7) to*

$$\begin{aligned}\{\tilde f,\tilde g\}_R^{(n)}=&-\frac{\alpha}{lmn}\int\frac{\delta f}{\delta\theta}\left(\partial^{-1}\frac{\delta g}{\delta\theta}\right)dx\\&+\int\mathrm{res}\left(\left(A\frac{\delta f}{\delta A}\right)_+A-A\left(\frac{\delta f}{\delta A}A)_+\right)\right)\frac{\delta g}{\delta A}dx,\\&+\frac{1}{l}\int\left(\mathrm{res}\left[A,\frac{\delta f}{\delta A}\right]\left(\partial^{-1}\mathrm{res}\left[A,\frac{\delta g}{\delta A}\right]\right)\right)dx\\&+\alpha\int\mathrm{res}\left(\left(B\frac{\delta f}{\delta B}\right)_+B-B\left(\frac{\delta f}{\delta B}B\right)_+\right)\frac{\delta g}{\delta B}dx,\\&+\frac{\alpha}{m}\int\left(\mathrm{res}\left[B,\frac{\delta f}{\delta B}\right]\left(\partial^{-1}\mathrm{res}\left[B,\frac{\delta g}{\delta B}\right]\right)\right)dx.\end{aligned}\tag{21}$$

Proof. Here we only show the proof for $\alpha=-1$, in this case $l=n+m$. The other case ($\alpha=1$) is easier. First we calculate $\delta\tilde f$ with respect to L and θ,A,B respectively. We have

$$\begin{aligned}\delta\tilde f=&\int\mathrm{res}\frac{\delta f}{\delta L}\delta Ldx\\=&\int\mathrm{res}\frac{\delta f}{\delta L}\left(-m\delta\theta L+\mathrm{e}^{-m\theta}\delta A\mathrm{e}^{-n\theta}B^{-1}\mathrm{e}^{l\theta}\right.\\&-n\mathrm{e}^{-m\theta}A\mathrm{e}^{-n\theta}\delta\theta B^{-1}\mathrm{e}^{l\theta}\\&-\mathrm{e}^{-m\theta}A\mathrm{e}^{-n\theta}B^{-1}\delta BB^{-1}\mathrm{e}^{l\theta}\\&\left.+l\mathrm{e}^{-m\theta}A\mathrm{e}^{-n\theta}B^{-1}\mathrm{e}^{l\theta}\delta\theta\right),\end{aligned}$$

on the one hand and

$$\delta\tilde f=\int\mathrm{res}\left(\frac{\delta f}{\delta A}\delta Adx+\frac{\delta f}{\delta B}\delta B+\frac{\delta f}{\delta\theta}\delta\theta\right)dx$$

on the other hand. So compare coefficients of δA, δB and $\delta\theta$ in the above two expressions, we find

$$\begin{aligned}\frac{\delta f}{\delta\theta}=&\mathrm{res}\left(-mL\frac{\delta f}{\delta L}+l\frac{\delta f}{\delta L}L-nB^{-1}\mathrm{e}^{l\theta}\frac{\delta f}{\delta L}\mathrm{e}^{-m\theta}A\mathrm{e}^{-n\theta}\right),\\\frac{\delta f}{\delta A}=&\mathrm{e}^{-n\theta}B^{-1}\mathrm{e}^{l\theta}\frac{\delta f}{\delta L}\mathrm{e}^{-m\theta}\qquad\mathrm{mod}\ R(-\infty,-l)\cup R(0,\infty),\\\frac{\delta f}{\delta B}=&-B^{-1}\mathrm{e}^{l\theta}\frac{\delta f}{\delta L}\mathrm{e}^{-m\theta}A\mathrm{e}^{-n\theta}B^{-1}\quad\mathrm{mod}\ R(-\infty,-m)\cup R(0,\infty).\end{aligned}\tag{22}$$

Then the substitution of them and similar expressions associated with $\tilde{g}$ into the right hand side of (21) completes our proof.

Notice that in (21), the second term and the third term form the second Poisson bracket associated with the l^{th} KdV hierarchy and the last two terms form the second Poisson bracket for the m^{th} KdV hierarchy, therefore the restriction decomposes the Poisson bracket. In terms of the fields J, v_j's and w_j's, this decomposition reads

$$\begin{aligned}
&\{J(x), J(y)\}_R^{(n)} = \frac{\alpha}{lmn}\delta'(x-y), \\
&\{v_i(x),\ v_j(y)\}_R^{(n)} = J_{i,j}^{(l)}(v)\delta(x-y), \\
&\{w_r(x),\ w_s(y)\}_R^{(n)} = \alpha J_{r,s}^{(m)}(w)\delta(x-y),
\end{aligned} \tag{23}$$

and these three groups of fields mutually commute, where $1 \le i, j \le l-2$ and $1 \le r, s \le m-2$. So we have

Proposition 3. *The $W(n,m)$-algebra and the W_n-algebra are isomorphic, respectively via a Miura map, to $W_l \oplus W_m \oplus U(1)$ algebra, where $l = n+m$, $m \ge 2$ for the isomorphism of $W(n,m)$-algebra, and $l = n-m$, $2 \le m \le n-2$ for the isomorphism of W_n-algebra.*

The isomorphism for $W(n,m)$-algebra was shown in [2] and the proof there depends on the derivation of a nontrivial Hamiltonian equation and the uniqueness of the second Hamiltonian structure of this equation. Here we have provided a more constructive and pure algebraical proof not only for $W(n,m)$-algebra but also for W_n-algebra.

It is very interest, although seems obviously, that the subalgebra $Vir[c(n)]$ in (12) also has a similar decomposition, i.e. $Vir[c(n)]$ is decomposed as a direct sum of the subalgebra $Vir[c(l)]$ of W_l and the sublagebra $Vir[c(m)]$ of W_m, as well as a Virasoro algebra $Vir[c_3]$ with central charge $c_3 = \alpha 3lmn$. We explain this decomposition for $\alpha = 1$, the other case is similar. By comparing coefficients on both sides of (2) (i.e. (18) with $\alpha = 1$), we have

$$u_{n-2} = v_{l-2} + w_{m-2} + \sigma, \tag{24}$$

where

$$\sigma = -\frac{lmn}{2}\left(J_x + J^2\right). \tag{25}$$

The fact can be shown immediately since v_{l-2} and w_{m-2} generate respectively the Virasoro algebras $Vir[c(l)]$ and $Vir[c(m)]$ and σ also generates a Virasoro $Vir[c_3]$ with the central charge $c_3 = 3lmn$.

As was pointed in [2], the fields J behaves like a gluon, which mediates the interaction between the W_m-algebra and W_{n+m}-algebra in the case of the $W(n,m)$-algebra. Here let us have a look of its behaviour from the viewpoint of the free field realization of the W_n-

algebra. Similar discussion is also valid in the case of $W(n,m)$-algebra. It is well-known that (see [11]) the factorization

$$L(\partial) = \partial^n + \sum_{j=0}^{n-2} u_j \partial^j = (\partial + p_1)(\partial + p_2)\cdots(\partial + p_n), \tag{26}$$

with

$$\sum_{j=1}^{n} p_j = 0, \tag{27}$$

leads to the Poisson brackets among the fields p_j

$$\{p_r(x), p_s(y)\}_R^{(n)} = (\delta_{rs} - \frac{1}{n})\delta'(x-y), \quad r,s = 1,2,\cdots,n. \tag{28}$$

The free field realization of W_n-algebra can be constructed by introducing $n-1$ free currents $j_1 = \varphi_1', \cdots, j_{n-1} = \varphi_{n-1}'$ and an overcomplete set of vectors $\vec{h}_r$, $r = 1,\cdots,n$ in $(n-1)$-dimensional Euclidean space with

$$\sum_{r=1}^{n} \vec{h}_r = 0, \quad \vec{h}_r \cdot \vec{h}_s = (\delta_{rs} - \frac{1}{n}). \tag{29}$$

Let $n = l+m$, $2 \le l, m \le n-2$, and introduce a group of new fields by

$$\begin{aligned} J &= \frac{1}{lm}\sum_{r=1}^{l} p_r = -\frac{1}{lm}\sum_{r=l+1}^{n} p_r, \\ \bar{p}_r &= p_r - mJ, \quad r = 1,\cdots,l, \\ \bar{p}_r &= p_r + lJ, \quad r = l+1,\cdots,n = l+m. \end{aligned} \tag{30}$$

Then we find that

$$\sum_{r=1}^{l} \bar{p}_r = \sum_{r=l+1}^{n} \bar{p}_r = 0. \tag{31}$$

The Poisson brackets among these new fields are

$$\begin{aligned} &\{J(x), J(y)\}_R^{(n)} = \frac{1}{lmn}\delta'(x-y), \\ &\{\bar{p}_r(x), \bar{p}_s(y)\}_R^{(n)} = (\delta_{rs} - \frac{1}{l})\delta'(x-y), \quad r,s = 1,\cdots,l, \\ &\{\bar{p}_k(x), \bar{p}_l(y)\}_R^{(n)} = (\delta_{kl} - \frac{1}{m})\delta'(x-y), \quad k,l = l+1,\cdots,n = l+m, \end{aligned} \tag{32}$$

and all other possible brackets vanish. So the field J decomposes the algebra (28) into two pieces of the same type of algebra.

Let

$$
\begin{aligned}
A(\partial) &= (\partial + \bar{p}_1) \cdots (\partial + \bar{p}_l), \\
B(\partial) &= (\partial + \bar{p}_{l+1}) \cdots (\partial + \bar{p}_n),
\end{aligned} \tag{33}
$$

they are in the form of (20). By using (30) we have the relation between L, and A and B

$$
L = A(\partial + mJ)B(\partial - lJ), \tag{34}
$$

which is the same as (2).

To study the Hamiltonian property of the equations for $J = \theta_x$, A and B, we first define

$$
\begin{aligned}
\hat{L} &= \mathrm{e}^{\alpha l\theta} B^{\alpha} \mathrm{e}^{-\alpha n\theta} A \mathrm{e}^{m\theta}, \\
\hat{P}_r &= \left(\hat{L}^{\frac{r}{n}}\right)_+,
\end{aligned} \tag{35}
$$

then we have

Proposition 4. *The restriction of $\tilde{h}_r(L)$ on the subspace of PDOs in the form of (18) is the Hamiltonian of the flow for $J = \theta_x, A, B$. The flow can be represented by the Lax-Manakov triad representation.*

$$
\begin{aligned}
\partial_{t_r} J &= \frac{1}{ml}\left(res\hat{L}^{\frac{r}{n}} - resL^{\frac{r}{n}}\right), \\
\partial_{t_r} A &= M_r A - A\hat{M}_r, \\
\partial_{t_r} B &= N_r B - B\hat{N}_r,
\end{aligned} \tag{36}
$$

where

$$
\begin{aligned}
M_r &= m\theta_{t_r} + e^{m\theta} P_r e^{-m\theta}, \\
\hat{M}_r &= m\theta_{t_r} + e^{m\theta} \hat{P}_r e^{-m\theta},
\end{aligned} \tag{37}
$$

and in the case of $\alpha = -1$, $l = n + m$:

$$
\begin{aligned}
N_r &= l\theta_{t_r} + e^{l\theta} P_r e^{-l\theta}, \\
\hat{N}_r &= l\theta_{t_r} + e^{l\theta} \hat{P}_r e^{-l\theta},
\end{aligned} \tag{38}
$$

while in the case of $\alpha = 1$, $l = n - m$:

$$
\begin{aligned}
N_r &= -l\theta_{t_r} + \mathrm{e}^{-l\theta} \hat{P}_r \mathrm{e}^{l\theta}, \\
\hat{N}_r &= -l\theta_{t_r} + \mathrm{e}^{-l\theta} P_r \mathrm{e}^{l\theta}.
\end{aligned} \tag{39}
$$

Proof. We also show the proof for $\alpha = -1$ and $l = n + m$. Because of (15) we have

$$
\frac{\delta h_r}{\delta\theta} = \mathrm{res}\left(nL^{\frac{r}{n}} - nB^{-1}\mathrm{e}^{l\theta} L^{\frac{r-n}{n}} \mathrm{e}^{-m\theta} A \mathrm{e}^{-n\theta}\right),
$$

the second term in the above can be written as

$$\begin{aligned}\left(B^{-1}\mathrm{e}^{l\theta}L^{\frac{r-n}{n}}\mathrm{e}^{-m\theta}A\mathrm{e}^{-n\theta}\right)^{n\frac{1}{n}} &= \left(B^{-1}\mathrm{e}^{l\theta}L^{r-1}\mathrm{e}^{-m\theta}A\mathrm{e}^{-n\theta}\right)^{\frac{1}{n}}\\ &= \left(\mathrm{e}^{l\theta}\hat{L}^{r}\mathrm{e}^{-l\theta}\right)^{\frac{1}{n}}\\ &= \mathrm{e}^{l\theta}\hat{L}^{\frac{r}{n}}\mathrm{e}^{-l\theta},\end{aligned}$$

so

$$\frac{\delta h_r}{\delta\theta} = n\left(\mathrm{res}L^{\frac{r}{n}} - \mathrm{res}\hat{L}^{\frac{r}{n}}\right).$$

The equation for θ is

$$\begin{aligned}\partial_{t_r}\theta &= \{\tilde{h}_r, \theta\}\\ &= -\frac{1}{lmn}\left(\partial^{-1}\frac{\delta h_r}{\delta\theta}\right),\end{aligned}$$

which immediately implies the first equation in (36).

To calculate the equation for A, we first have

$$A\frac{\delta h_r}{\delta A} = \mathrm{e}^{m\theta}L^{\frac{r}{n}}\mathrm{e}^{-m\theta},$$

and

$$\begin{aligned}\frac{\delta h_r}{\delta A}A &= \mathrm{e}^{-n\theta}B^{-1}\mathrm{e}^{l\theta}L^{\frac{r-n}{n}}\mathrm{e}^{-m\theta}A\\ &= \left(\mathrm{e}^{-n\theta}B^{-1}\mathrm{e}^{l\theta}L^{\frac{r-n}{n}}\mathrm{e}^{-m\theta}A\right)^{n\frac{1}{n}}\\ &= \left(\mathrm{e}^{-n\theta}B^{-1}\mathrm{e}^{l\theta}L^{r-1}\mathrm{e}^{-m\theta}A\right)^{\frac{1}{n}}\\ &= \left(\mathrm{e}^{m\theta}\hat{L}^{r}\mathrm{e}^{-m\theta}\right)^{\frac{1}{n}} = \mathrm{e}^{m\theta}\hat{L}^{\frac{r}{n}}\mathrm{e}^{-m\theta},\end{aligned}$$

so

$$\mathrm{res}\left[A, \frac{\delta h_r}{\delta A}\right] = \mathrm{res}L^{\frac{r}{n}} - \mathrm{res}\hat{L}^{\frac{r}{n}} = -lm\theta_{xt_r}.$$

The equation for A now reads

$$\begin{aligned}\partial_{t_r}A =& \{h_r, A\} = \sum_{j=0}^{l-2}\{h_r, v_j\}\partial^j\\ =& \left(A\frac{\delta h_r}{\delta A}\right)_+ A - A\left(\frac{\delta h_r}{\delta A}A\right)_+ - ml\theta_{t_r,x}\partial^{l-1}\\ &+ m\sum_{j=0}^{l-2}\left(\int\left(\theta_{t_r}\mathrm{res}\left[A, \frac{\delta v_j}{\delta A}\right]\right)dx'\right)\partial^j,\end{aligned}$$

where $\delta v_j/\delta A = \partial^{-j-1}\delta(x'-x)$. The last two terms in the above is $m\left(\theta_{t_r}A - A\theta_{t_r}\right)$, so we have the second equation in (36) and similarly the third one.

Let us show the first nontrivial flow ($r=2$) with $\alpha=\pm1$ and $l=n-\alpha m$

$$
\begin{aligned}
\partial_{t_2} J =& \alpha\left(\frac{2}{nl}v-\frac{2}{nm}w+(l-\alpha m)J^2\right)_x,\\
\partial_{t_2} A =& \left((\partial-mJ)^2+\frac{2}{l}v-mlJ_x-m^2J^2\right)A\\
&-A\left((\partial-mJ)^2+\frac{2}{l}v+mlJ_x-m^2J^2\right),\\
\partial_{t_2} B =& \left((\partial+\alpha lJ)^2+\frac{2}{m}w+\alpha mlJ_x-l^2J^2\right)B\\
&-B\left((\partial+\alpha lJ)^2+\frac{2}{m}w-\alpha mlJ_x-l^2J^2\right),
\end{aligned}
\tag{40}
$$

where we have rename v_{l-2} by v and w_{m-2} by w. In the case of $\alpha=-1$ and $l=n+m$, the above equation is exactly the same as the equation shown in [2].

Proposition 5. *Define*

$$
\tilde{J}=\int J dx, \tag{41}
$$

then it is the conserved quantity of the hierarchy in (36), and commutes with other conserved quantities $\tilde{h}_r$.

Proof: First using the relation

$$
\hat{L}=\mathrm{e}^{-m\theta}A^{-1}\mathrm{e}^{m\theta}L\mathrm{e}^{-m\theta}A\mathrm{e}^{m\theta},
$$

for L in (18) and $\hat{L}$ in (35), then the identity $\int \mathrm{res}PQdx=\int \mathrm{res}QPdx$ for any PDOs P and Q immediately implies that

$$
\frac{\partial\tilde{J}}{\partial t_r}=\frac{\partial}{\partial t_r}\int Jdx=0.
$$

On the other hand one can check that

$$
\{\tilde{h}_r,\tilde{J}\}_R^{(n)}=0,
$$

for the Poisson bracket in (21).

Remark. It can easily checked that both L in (18) and $\hat{L}$ in (35) satisfy the KP hierarchy when $\alpha=-1$ and n^{th} KdV hierarchy when $\alpha=1$. In particular when $\alpha=1$, equations in (36) are type of modified ones of the n^{th} KdV hierarchy.

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